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## Eigengaps for hub-dominant matrices

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Hub-dominant matrices are natural extensions of hub matrices. In this article we study eigengaps of the Gram matrix associated with a hub-dominant matrix. A class of hub-dominant matrices is then constructed by using equiangular tight frames.

**Keywords:** hub matrices; eigengaps; equiangular tight frames; arrowhead matrices

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### 1. Introduction

The notion of hub matrices was first proposed by Kung and Suter in [1]. A matrix is called a hub matrix if, without loss of generality, the Euclidean norm of its last column (called hub column) is greater than that of the rest columns (non-hub columns) and, in addition, all non-hub columns are orthogonal to each other with respect to the Euclidean inner product. The framework of hub matrices can be used to describe a variety of wireless communication systems. For example, the hub matrix theory was applied to beamforming MIMO communication systems in [1]. The eigenstructure of the Gram matrices of the corresponding hub matrices was exploited to analyse the performance and capacity of the systems. Here, the eigenstructure of a positive semi-definite symmetric matrix refers to the ratio of its two leading eigenvalues. One mathematical issue in hub matrix theory is to infer the eigenstructure of the Gram matrix of a given hub matrix through the Euclidean norms of all columns of the hub matrix and the Euclidean inner products between the non-hub columns and the hub column. For instance, the eigenstructure of the Gram matrix of a hub matrix was characterized by the hub column and a non-hub column with the largest Euclidean norm in [1]. In our recent work [2], an improved estimate on the eigenstructure has been achieved by fully exploiting information from the hub column and all non-hub columns.

The Gram matrix of a hub matrix is a so-called arrowhead matrix whose all principal minors except itself are diagonal matrices. In other words, the Gram

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matrix can be obtained by bordering a diagonal matrix with one column from the bottom and one row from the right. This diagonal matrix can be extended to other types of matrices such as block diagonal matrices and diagonal dominant matrices. Actually, when the diagonal matrix is replaced by a diagonal dominant matrix, the corresponding hub matrix becomes a hub-dominant matrix (see [1]). Hub-dominant matrices can be used in areas such as distributed beamforming and power control in wireless *ad hoc* networks [1]. The eigenstructure of the Gram matrix of a hub-dominant matrix has been studied in [1]. In this article, we will improve the estimate of the eigenstructure based on our recent work [2]. We will also construct and analyse a class of hub-dominant matrices via equiangular tight frames.

This article is organized as follows. In Section 2, we give the definition of hub-dominant matrices and review the result on the lower and upper bounds of the Gram matrix of a hub-dominant matrix in [1]. Few remarks on the result are made to motivate our current work. In Section 3 we develop some useful lemmas for estimating eigenvalues of positive semi-definite symmetric matrices. In Section 4, we present our new results on the lower and upper bounds for the eigengap of the Gram matrix of a hub-dominant matrix. The improvement of the bounds over the result given in [1] is discussed. In Section 5, we construct a class of hub-dominant matrices through equiangular tight frames. Our conclusion is drawn in Section 6.

## 2. Hub and hub-dominant matrices

In the original work of [1], the definition of hub matrices was introduced first, and then extended to a general case of hub-dominant matrices. The present presentation differs in that. We first give the definition of hub-dominant matrices and then treat a hub matrix as a special kind of hub-dominant matrix.

*Definition 1* A matrix  $A \in \mathbb{R}^{n \times m}$  is called a candidate-hub-dominant matrix if its first  $m-1$  columns  $a_i$ ,  $i = 1, \dots, m-1$ , satisfy the following conditions:

$$\xi \|a_i\|^2 \leq \sum_{j=1}^{m-1} |\langle a_i, a_j \rangle| \leq \eta \|a_i\|^2 \quad \text{for } i = 1, \dots, m-1, \quad (1)$$

where  $\xi \geq 1$  and  $\eta \leq 2$ . If, in addition, the last column has its Euclidean norm greater than or equal to that of any other column, then  $A$  is called a hub-dominant matrix and the last column is called the hub column.

For a given hub-dominant matrix  $A$ , its Gram matrix is denoted by  $Q := A^t A$  and called the system matrix associated with the matrix  $A$ . Assume that  $A = [a_1 \ a_2 \ \dots \ a_{m-1} \ a_m]$  is a hub-dominant matrix. The corresponding system matrix  $Q$  is partitioned into a form as follows:

$$Q = A^t A = \begin{bmatrix} D & c \\ c^t & b \end{bmatrix}, \quad (2)$$

where the matrix  $D$ , the vector  $c$  and the real number  $b$  in (2) are

$$D = \begin{bmatrix} \|a_1\|^2 & \langle a_1, a_2 \rangle & \cdots & \langle a_1, a_{m-1} \rangle \\ \langle a_2, a_1 \rangle & \|a_2\|^2 & \cdots & \langle a_2, a_{m-1} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle a_{m-1}, a_1 \rangle & \langle a_{m-1}, a_2 \rangle & \cdots & \|a_{m-1}\|^2 \end{bmatrix}, \quad c = \begin{bmatrix} \langle a_1, a_m \rangle \\ \langle a_2, a_m \rangle \\ \vdots \\ \langle a_{m-1}, a_m \rangle \end{bmatrix}, \quad b = \|a_m\|^2, \quad (3)$$

respectively. The matrix  $D$  represents the correlation among the non-hub nodes. The inequality (1) implies  $D$  being a diagonally dominant matrix. This is the reason why  $A$  is named a diagonally dominant hub matrix. The vector  $c$  reflects the correlation between the non-hub nodes and the hub node.

When the parameters  $\xi$  and  $\eta$  in (1) are identical and both equal to 1, we have  $\sum_{j=1}^{m-1} |\langle a_i, a_j \rangle| = \|a_i\|^2$ , which imply that

$$\langle a_i, a_j \rangle = 0 \quad \text{for all } 1 \leq i < j \leq m-1.$$

This means that all non-hub columns are orthogonal to each other with respect to the Euclidean inner product. In such case, the matrix  $D$  in (2) and (3) reduces to a diagonal matrix with  $\|a_1\|^2, \|a_2\|^2, \dots, \|a_{m-1}\|^2$  as its diagonal entries. The matrix  $A$  is called a *hub matrix* [1] and the corresponding  $Q$  is an arrowhead matrix [3,4].

It is of particular interest in estimating the eigenvalues of the system matrix associated with a hub-dominant matrix, mostly, the ratio of leading two eigenvalues of the system matrix via the elements of the hub-dominant matrix [1]. To this end, we introduce definitions of hub-gaps for a hub-dominant matrix and eigengaps for the associated system matrix.

**Definition 2** (hub-gap and eigengap) Let  $A \in \mathbb{R}^{n \times m}$  be a matrix with its columns denoted by  $a_1, \dots, a_m$  arranged in increasing order. For  $i=1, \dots, m-1$ , the  $i$ -th hub-gap of  $A$  is defined to be

$$\text{HG}_i(A) := \frac{\|a_{m-i+1}\|^2}{\|a_{m-i}\|^2}.$$

Let  $Q$  be the associated system matrix with eigenvalues denoted by  $\lambda_1, \dots, \lambda_m$  with  $\lambda_1 \leq \dots \leq \lambda_m$ . For  $i=1, \dots, m-1$ , the  $i$ -th eigengap of  $Q$  is defined to be

$$\text{EG}_i(Q) := \frac{\lambda_{m-i+1}}{\lambda_{m-i}}.$$

With this notation, an estimate of the eigengap of the system matrix  $Q$  was given in [1] in terms of the hub-gaps of the corresponding hub-dominant matrix. We state the estimate in the following theorem.

**THEOREM 1** [7] Let  $A \in \mathbb{R}^{n \times m}$  be a hub-dominant matrix with its columns denoted by  $a_1, a_2, \dots, a_m$  arranged in increasing order. Let  $Q = A^t A \in \mathbb{R}^{m \times m}$  be a corresponding system matrix with its eigenvalues denoted by  $\lambda_1, \dots, \lambda_m$  with  $\lambda_1 \leq \dots \leq \lambda_m$ . Let  $d_{ii}$  and  $\sigma_i$  denote the diagonal entry and the sum of the magnitudes of off-diagonal entries, respectively, in row  $i$  of  $D$  for  $i=1, \dots, m-1$ . Then

$$\frac{1}{2} \text{HG}_1(A) \leq \text{EG}_1(Q) \leq \frac{d_{(m-1)(m-1)} + b + \sum_{i=1}^{m-2} \sigma_i}{d_{(m-2)(m-2)} - \sigma_{m-2}}. \quad (4)$$

We make three remarks on this theorem.

*Remark 1* We would like to take this opportunity to emphasize that the difference  $d_{(m-2)(m-2)} - \sigma_{m-2}$  in the denominator of the upper bound shown in (4) should be understood as the second largest number in the sequence  $d_{11} - \sigma_1, d_{22} - \sigma_2, \dots, d_{(m-1)(m-1)} - \sigma_{m-1}$ .

*Remark 2* The expression in (4) fails to give an upper bound of  $\text{EG}_1(Q)$  when  $d_{(m-2)(m-2)} - \sigma_{m-2} = 0$ .

*Remark 3* When  $A$  is a hub matrix, (4) becomes

$$\frac{1}{2} \text{HG}_1(A) \leq \text{EG}_1(Q) \leq (\text{HG}_1(A) + 1) \text{HG}_2(A).$$

In comparison to the lower and upper bounds of  $\text{EG}_1(Q)$  given by Theorem 4 in [1], we see that the upper bound of  $\text{EG}_1(Q)$  in [1] is the same as the one in the above inequality, but the lower bound doubles. In this sense there is room to further improve the lower bound of  $\text{EG}_1(Q)$  in (4).

These remarks partially motivated us to conduct our current work in this article. Since the eigengap  $\text{EG}_1(Q)$  is defined as the ratio of leading two largest eigenvalues of  $Q$ , it is desirable to estimate the leading two largest eigenvalues of  $Q$  in terms of the entries of the associated hub-dominant matrix  $A$  with a high accuracy in order to have tighter lower and upper bounds of  $\text{EG}_1(Q)$ . This will be our main task in the next section.

### 3. Some lemmas

Let  $A$  be a hub-dominant matrix and  $Q$  be the associated Gram matrix or the system matrix. The matrix  $Q$  has a form of (2), which is partitioned according to the non-hub columns and the hub column of  $A$ . We will use this structure of  $Q$  to estimate its eigenvalues.

When  $A$  is a hub matrix,  $Q$  is an arrowhead matrix. The Cauchy interlacing theorem is usually adopted to estimate the eigenvalues of an arrowhead matrix [5]. In [2], we provide new estimates for the eigenvalues of arrowhead matrices. It was proved there that the estimated lower and upper bounds of the eigenvalues are tighter than that by using the Cauchy interlacing theorem. In the meanwhile, it was numerically examined that the estimates for the least and largest eigenvalues are better than that based on a method in [6]. Because of this, we propose to apply the results in [2] for new estimates of the eigenvalues of  $Q$  when  $A$  is a hub-dominant matrix.

The approach we adopt is motivated from the structure of  $Q$  in (2). By looking at it, when the submatrix  $D$  in  $Q$  is diagonal,  $Q$  is an arrowhead matrix. Therefore the results in [2] can be directly applicable in such situation. Fortunately, if the submatrix  $D$  is not a diagonal,  $D$  is unitarily similar to a diagonal matrix  $\Gamma$ , i.e. there exists an orthogonal matrix  $U$  such that

$$D = U^t \Gamma U,$$

where  $\Gamma = \text{diag}(\mu_1, \mu_2, \dots, \mu_{m-1})$  with  $\mu_1 \leq \mu_2 \leq \dots \leq \mu_{m-1}$ . Hence, we have

$$\begin{bmatrix} \Gamma & z \\ z^t & b \end{bmatrix} = \begin{bmatrix} U & \\ & 1 \end{bmatrix} \begin{bmatrix} D & c \\ c^t & b \end{bmatrix} \begin{bmatrix} U^t & \\ & 1 \end{bmatrix}$$

where  $z = Uc$ . Therefore, the eigenvalues of the matrix  $Q$  are the same as the arrowhead matrix

$$B := \begin{bmatrix} \Gamma & z \\ z^t & b \end{bmatrix}. \quad (5)$$

We call  $B$  an arrowhead matrix induced from the system matrix  $Q$ .

In what follows, the eigenvalues  $\lambda_i$  of a symmetric matrix  $P \in \mathbb{R}^{n \times n}$  are always ordered such that  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ . When the dependence of the eigenvalues on the matrix  $P$  needs to be determined, we simply write  $\lambda_1(P) \leq \lambda_2(P) \leq \dots \leq \lambda_n(P)$  instead.

Next, we present the bounds for the eigenvalues of the arrowhead matrix  $B$  developed in [2].

LEMMA 1 [11] Let  $B \in \mathbb{R}^{m \times m}$  be an arrowhead matrix in (5) having  $\Gamma = \text{diag}(\mu_1, \dots, \mu_{m-1})$  with  $\mu_1 \leq \mu_2 \leq \dots \leq \mu_{m-1} \leq b$ . Then

$$\lambda_j(B) \leq \begin{cases} \min\{\mu_1, f_-(\frac{\mu_{m-1}}{b})\} & \text{if } j = 1; \\ \mu_j & \text{if } 2 \leq j \leq m-1; \\ f_+(\frac{\mu_{m-1}}{b}) & \text{if } j = m \end{cases} \quad (6)$$

and

$$\lambda_j(B) \geq \begin{cases} f_-(\frac{\mu_1}{b}) & \text{if } j = 1; \\ \max\{\mu_{j-1}, f_-(\frac{\mu_j}{b})\} & \text{if } 2 \leq j \leq m-1; \\ f_+(\frac{\mu_1}{b}) & \text{if } j = m, \end{cases} \quad (7)$$

where

$$f_{\pm}(\rho) = \rho b + \frac{1}{2} \left( (1 - \rho)b \pm \sqrt{(1 - \rho)^2 b^2 + 4\|z\|^2} \right).$$

As a direct application of Lemma 1, we have the following result on the bounds of the eigengap of the system matrix  $Q$ .

THEOREM 2 Let  $A \in \mathbb{R}^{n \times m}$  be a hub-dominant matrix with its columns denoted by  $a_1, \dots, a_m$  arranged in increasing order. Let  $Q = A^t A \in \mathbb{R}^{m \times m}$  in (2) be the corresponding system matrix and  $B$  in (5) be the arrowhead matrix induced from  $Q$ . Then we have

$$\frac{f_+(\mu_1/b)}{\mu_{m-1}} \leq \text{EG}_1(Q) \leq \frac{f_+(\mu_{m-1}/b)}{\max\{\mu_{m-2}, f_-(\mu_{m-1}/b)\}}, \quad (8)$$

where  $\mu_1, \mu_2, \dots, \mu_{m-1}$  arranged in increasing order are the diagonal elements of the diagonal matrix  $\Gamma$  in  $B$ .

COROLLARY 1 Let  $A \in \mathbb{R}^{n \times m}$  be a hub-dominant matrix with its columns denoted by  $a_1, \dots, a_m$  and arranged in increasing order. Suppose that all non-hub columns  $a_1, a_2, \dots, a_{m-1}$  are orthogonal to each other with respect to the Euclidean inner product. Then the lower and upper bounds of  $\text{EG}_1(Q)$  in Theorem 2 are sharper than those in Theorem 1.

*Proof* When all non-hub columns  $a_1, a_2, \dots, a_{m-1}$  are orthogonal to each other with respect to the Euclidean inner product, we know that

$$\mu_i = d_{ii} = \|a_{ii}\|^2, \quad \text{for } i = 1, \dots, m-1.$$

In this case,

$$\frac{f_+(\mu_1/b)}{\mu_{m-1}} = \frac{(b + d_{11}) + \sqrt{(b - d_{11})^2 + 4\|c\|^2}}{2d_{(m-1)(m-1)}} \geq \frac{b}{d_{(m-1)(m-1)}} = \text{HG}_1(A),$$

that is, the lower bound of  $\text{EG}_1(Q)$  in Theorem 2 is sharper than that in Theorem 1.

To show the upper bound of  $\text{EG}_1(Q)$  in Theorem 2 is sharper than that in Theorem 1, i.e.

$$\frac{f_+(d_{(m-1)(m-1)}/b)}{\max\{d_{(m-1)(m-1)}, f_-(d_{(m-1)(m-1)}/b)\}} \leq \frac{d_{(m-1)(m-1)} + b}{d_{(m-1)(m-1)}},$$

we need to show that

$$f_+(d_{(m-1)(m-1)}/b) \leq d_{(m-1)(m-1)} + b. \quad (9)$$

It is equivalent to showing

$$\|c\|^2 \leq d_{(m-1)(m-1)}b.$$

The above inequality is true because

$$\begin{aligned} \|c\|^2 &= \sum_{i=1}^{m-1} |\langle a_i, a_m \rangle|^2 = \sum_{i=1}^{m-1} \frac{\|a_i\|^2}{\|a_i\|^2} |\langle a_i, a_m \rangle|^2 \\ &\leq \|a_{m-1}\|^2 \sum_{i=1}^{m-1} \frac{1}{\|a_i\|^2} |\langle a_i, a_m \rangle|^2 \\ &\leq \|a_{m-1}\|^2 \|a_m\|^2 = d_{(m-1)(m-1)}b. \end{aligned}$$

This completes the proof. ■

From Corollary 1, we have tighter bounds for the eigengap  $\text{EG}_1(Q)$  than that in Theorem 1 when  $A$  is a hub matrix. For  $A$  being a hub-dominant matrix, the bounds given in Theorem 2 involve the smallest eigenvalue  $\mu_1$  and two largest eigenvalues  $\mu_{m-2}$  and  $\mu_{m-1}$  of  $D$  which are not available in general. Hence, we need to estimate these eigenvalues in terms of entries of the associated hub-dominant matrix  $A$  explicitly. To this end, for the eigenvalues  $\mu_1$  and  $\mu_{m-2}$ , we need to give lower bounds of them; for the eigenvalue  $\mu_{m-1}$ , we need both lower and upper bounds.

The following lemmas are useful in estimating eigenvalues  $\mu_1$ ,  $\mu_{m-2}$  and  $\mu_{m-1}$  of  $D$ .

**LEMMA 2** [4, p. 191] *Let  $P \in \mathbb{R}^{n \times n}$  be a symmetric matrix, let  $r$  be an integer with  $1 \leq r \leq n$ , and let  $P_r$  denote any  $r \times r$  principal submatrix of  $P$  (obtaining by deleting  $n-r$  rows and the corresponding columns from  $P$ ). For each integer  $k$  such that  $1 \leq k \leq r$  we have*

$$\lambda_k(P) \leq \lambda_k(P_r) \leq \lambda_{k+n-r}(P).$$

This lemma shows that the eigenvalues of  $P$  can be estimated through eigenvalues of its principal submatrices. To make this lemma useful, the key is to identify principal submatrices of  $P$  whose eigenvalues are easily computed. Clearly, we can explicitly calculate the eigenvalues of the principal submatrices  $P_r$  for  $r=1$  and 2. This leads to the following result.

LEMMA 3 *Let  $P \in \mathbb{R}^{n \times n}$  be a positive semi-definite symmetric matrix. Then*

- (1)  $\lambda_1(P) \leq \min_{1 \leq i \leq n} p_{ii}$  and  $\lambda_n(P) \geq \max_{1 \leq i \leq n} p_{ii}$ ,
- (2)  $\lambda_{n-1}(P) \geq \max_{1 \leq i < j \leq n} \frac{1}{2}((p_{ii} + p_{jj}) - \sqrt{(p_{ii} - p_{jj})^2 + 4p_{ij}^2})$ ,
- (3)  $\lambda_n(P) \geq \max_{1 \leq i < j \leq n} \frac{1}{2}((p_{ii} + p_{jj}) + \sqrt{(p_{ii} - p_{jj})^2 + 4p_{ij}^2})$ .

*Proof* The first item is a well-known result. It is a direct consequence of Lemma 2 by setting  $r=1$ . To prove the results in the second and third items, we set  $r=2$  in Lemma 2. The corresponding matrix  $P_2$  has the following form:

$$P_2 = \begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix},$$

where  $i$  and  $j$  are two distinct integers between 1 and  $n$ . Notice that all diagonal elements of  $P$  are non-negative. The smallest and largest eigenvalues of  $P_2$  are

$$\frac{1}{2} \left( (p_{ii} + p_{jj}) - \sqrt{(p_{ii} - p_{jj})^2 + 4p_{ij}^2} \right) \quad \text{and} \quad \frac{1}{2} \left( (p_{ii} + p_{jj}) + \sqrt{(p_{ii} - p_{jj})^2 + 4p_{ij}^2} \right),$$

respectively. The results stated in items 2 and 3 are followed from Lemma 2. ■

Since  $\frac{1}{2}((p_{ii} + p_{jj}) + \sqrt{(p_{ii} - p_{jj})^2 + 4p_{ij}^2}) \geq \max\{p_{ii}, p_{jj}\}$ , the lower bound for  $\lambda_n(P)$  from item 3 of Lemma 3 is tighter than that from item 1.

LEMMA 4 *Let  $P \in \mathbb{R}^{n \times n}$  be a diagonally dominant and positive semi-definite symmetric matrix. Let us denote  $\sigma_i$  the sum of the magnitudes of the off-diagonal elements of the  $i$ -th row of  $P$ , i.e.  $\sigma_i = \sum_{j=1, j \neq i}^n |p_{ij}|$ . Let us define*

$$X := \text{diag}(p_{11} - \sigma_1, p_{22} - \sigma_2, \dots, p_{nn} - \sigma_n).$$

*Then*

$$\lambda_i(P) \geq \lambda_i(X) \tag{10}$$

*for  $i=1, 2, \dots, n$ .*

*Proof* Since  $P$  is a diagonally dominant and positive semi-definite symmetric matrix, then all diagonal entries of  $X$  are non-negative. Define

$$Y := P - X.$$

Clearly,  $Y$  is a diagonally dominant symmetric matrix and its diagonal elements are  $\sigma_i$  which are non-negative for  $i=1, 2, \dots, n$ . Therefore,  $Y$  is a diagonally dominant and positive semi-definite symmetric matrix, hence all eigenvalues of  $Y$  are non-negative. Since  $P$  is a sum of the diagonal matrix  $X$  and the non-negative symmetric matrix  $Y$ , Weyl's monotonicity theorem (see, e.g. [7, page 63] or [4, page 182]) implies (10). ■



Both Lemmas 3 and 4 provide a lower bound for  $\lambda_n(P)$ . Since

$$\max\{p_{ii} - \sigma_i: i = 1, 2, \dots, n\} \leq p_{nn} \leq \frac{1}{2} \left( (p_{nn} + p_{jj}) + \sqrt{(p_{nn} - p_{jj})^2 + 4p_{nj}^2} \right)$$

for all  $j = 1, \dots, n$ , we choose the lower bound of  $\lambda_n(P)$  provided by Lemma 3.

Regarding the lower bound of  $\lambda_{n-1}(P)$  based on Lemmas 3 and 4, we have the following result.

**LEMMA 5** *Let  $P \in \mathbb{R}^{n \times n}$  be a symmetric diagonally dominant and positive semi-definite matrix. Let us denote  $\sigma_i$  the sum of the magnitudes of the off-diagonal elements of the  $i$ -th row of  $P$ , i.e.  $\sigma_i = \sum_{j=1, j \neq i}^n |p_{ij}|$ . Then the lower bound of  $\lambda_{n-1}(P)$  given in Lemma 3 is tighter than that in Lemma 4.*

*Proof* We assume that the diagonal entries  $p_{ii}$  of  $P$  have been arranged in increasing order. We consider two different cases. In the first case, we assume that

$$p_{nn} - \sigma_n = \max\{p_{ii} - \sigma_i: i = 1, 2, \dots, n\},$$

then

$$\max_{1 \leq i < j \leq n} \frac{1}{2} \left( (p_{ii} + p_{jj}) - \sqrt{(p_{ii} - p_{jj})^2 + 4p_{ij}^2} \right) \geq \max_{1 \leq i < j \leq n} (p_{ii} - |p_{ij}|) \geq \max_{1 \leq i \leq n-1} (p_{ii} - \sigma_i),$$

where  $\max_{1 \leq i \leq n-1} (p_{ii} - \sigma_i)$  is the second largest number among  $p_{11} - \sigma_1, p_{22} - \sigma_2, \dots, p_{nn} - \sigma_n$ , therefore, is the exact lower bound of  $\lambda_{n-1}(P)$  provided by Lemma 4.

In the second case, we assume that for an integer  $1 \leq j_0 \leq n-1$ ,

$$p_{j_0 j_0} - \sigma_{j_0} = \max\{p_{ii} - \sigma_i: i = 1, 2, \dots, n\},$$

then

$$\begin{aligned} \max_{1 \leq i < j \leq n} \frac{1}{2} \left( (p_{ii} + p_{jj}) - \sqrt{(p_{ii} - p_{jj})^2 + 4p_{ij}^2} \right) &\geq \max_{1 \leq i < j \leq j_0+1} \frac{1}{2} \left( (p_{ii} + p_{jj}) - \sqrt{(p_{ii} - p_{jj})^2 + 4p_{ij}^2} \right) \\ &\geq p_{j_0 j_0} - |p_{j_0(j_0+1)}| \\ &\geq p_{j_0 j_0} - \sigma_{j_0} = \max\{p_{ii} - \sigma_i: i = 1, 2, \dots, n\}. \end{aligned}$$

From both cases, we conclude that the statement of this lemma is true. ■

**LEMMA 6** *Let  $P \in \mathbb{R}^{n \times n}$  be a diagonally dominant and positive semi-definite symmetric matrix. Then*

$$\lambda_n(P) \leq \max_{1 \leq i \leq n} \sum_{j=1}^n |p_{ij}| \leq 2 \max_{1 \leq i \leq n} |p_{ii}|.$$

*Proof* Note that all eigenvalues of the positive semi-definite symmetric matrix  $P$  are real and non-negative. The first inequality is due to Gershgorin's disc theorem while the second inequality is due to  $P$  being a diagonally dominant matrix. ■

**LEMMA 7** *Let  $Q \in \mathbb{R}^{m \times m}$  be a positive semi-definite symmetric matrix. We partition  $Q$  as follows:*

$$Q = \begin{bmatrix} D & c \\ c^t & b \end{bmatrix},$$

where  $D \in \mathbb{R}^{(m-1) \times (m-1)}$  and  $c \in \mathbb{R}^{m-1}$ . Let us denote  $\sigma_i$  the sum of the magnitudes of the off-diagonal elements of the  $i$ -th row of  $D$ , i.e.  $\sigma_i := \sum_{j=1, j \neq i}^{m-1} |d_{ij}|$ , where  $d_{ij}$  is the  $(i, j)$ -th entry of  $D$  (of  $Q$  as well). Assume that  $D$  is a diagonally dominant matrix, then

$$\lambda_m(Q) \leq b + \max_{1 \leq i \leq m-1} \{d_{ii} + \sigma_i\}.$$

*Proof* As we know, from the beginning of this section, the matrix  $Q$  can be converted to an arrowhead matrix  $B$  given in (5) by a similarity transformation. Hence, the eigenvalues of  $Q$  and  $B$  are the same. Further, by the fact that the trace of a matrix is equal to the sum of its eigenvalues, we have

$$\mu_1 + \mu_2 + \cdots + \mu_{m-1} + b = \lambda_1(Q) + \lambda_2(Q) + \cdots + \lambda_m(Q), \quad (11)$$

where  $\mu_1, \mu_2, \dots, \mu_{m-1}$  arranged in increasing order are the eigenvalues of the diagonal matrix  $\Gamma$  in (5).

Applying the Cauchy interlacing theorem for  $B$  yields

$$\lambda_j(Q) \leq \mu_j \leq \lambda_{j+1}(Q) \quad (12)$$

for  $j = 1, 2, \dots, m-1$ . Applying (12) to (11) leads to

$$\mu_1 + \mu_2 + \cdots + \mu_{m-1} + b \geq \lambda_1(Q) + \mu_1 + \cdots + \mu_{m-2} + \lambda_m(Q).$$

Hence,

$$\lambda_m(Q) \leq b + \mu_{m-1} - \lambda_1(Q).$$

Since  $\lambda_1(Q) \geq 0$  and  $D$  is a diagonally dominant matrix, the result of this lemma follows from the above inequality together with Lemma 6.  $\blacksquare$

#### 4. Lower and upper bounds of the eigengap $\text{EG}_1(Q)$

In this section, we utilize the results given in the previous section to provide new lower and upper bounds for  $\text{EG}_1(Q)$ , where  $Q$  is the system matrix associated with a hub-dominant matrix  $A$ . We further show that the new bounds are tighter than that provided by Theorem 1.

**THEOREM 3** Let  $A \in \mathbb{R}^{n \times m}$  be the hub-dominant matrix having  $a_1, a_2, \dots, a_m$  as its columns with  $\|a_1\| \leq \|a_2\| \leq \cdots \leq \|a_m\|$ . Let  $Q = A^t A \in \mathbb{R}^{m \times m}$  in (2) be the corresponding system matrix. Let us denote  $\sigma_i := \sum_{j=1, j \neq i}^{m-1} |d_{ij}|$  as the sum of the magnitudes of the off-diagonal elements of the  $i$ -th row of  $D = [d_{ij}]_{i,j=1}^{m-1}$ . Then

$$\frac{f_+(\underline{s}/b)}{\max_{1 \leq i \leq m-1} \sum_{j=1}^{m-1} |d_{ij}|} \leq \text{EG}_1(Q), \quad (13)$$

where  $\underline{s} = \min\{d_{ii} - \sigma_i; i = 1, \dots, m-1\}$ .

*Proof* Let  $B$  in (5) be the reduced arrowhead matrix of  $Q$  and let  $\mu_1, \mu_2, \dots, \mu_{m-1}$  be the diagonal elements of the diagonal matrix  $\Gamma$  in  $B$  which have been arranged in increasing order. From Theorem 2, we have

$$\text{EG}_1(Q) \geq \frac{f_+(\mu_1/b)}{\mu_{m-1}}.$$

By Lemma 4, we have  $\mu_1 \geq \underline{s}$ . Since  $f_+$  is an increasing function, we get  $f_+(\mu_1/b) \geq f_+(\underline{s}/b)$ . By Lemma 6, we have  $\mu_{m-1} \leq \max_{1 \leq i \leq m-1} \sum_{j=1}^{m-1} |d_{ij}|$ . Hence

$$\text{EG}_1(Q) \geq \frac{f_+(\mu_1/b)}{\mu_{m-1}} \geq \frac{f_+(\underline{s}/b)}{\max_{1 \leq i \leq m-1} \sum_{j=1}^{m-1} |d_{ij}|}.$$

This shows that (13) holds. ■

**COROLLARY 2** *Under the same conditions as in Theorem 3, we have*

$$\frac{1}{4} \left( \text{HG}_1(A) + \sqrt{\text{HG}_1^2(A) + \frac{4\|c\|^2}{d_{(m-1)(m-1)}^2}} \right) \leq \text{EG}_1(Q). \quad (14)$$

*Proof* Since  $A \in \mathbb{R}^{n \times m}$  is a hub-dominant matrix, the submatrix  $D$  of  $Q$  in (2) is diagonally dominated. Hence  $\underline{s} \geq 0$  and

$$f_+(\underline{s}/b) \geq f_+(0) = \frac{1}{2} \left( b + \sqrt{b^2 + 4\|z\|^2} \right) = \frac{1}{2} \left( b + \sqrt{b^2 + 4\|c\|^2} \right).$$

The fact  $\|z\| = \|c\|$  is used above. Since  $D$  is a diagonally dominated matrix, we get

$$\max_{1 \leq i \leq m-1} \sum_{j=1}^{m-1} |d_{ij}| \leq \max_{1 \leq i \leq m-1} 2d_{ii} = 2d_{(m-1)(m-1)}.$$

Hence,

$$\frac{f_+(\underline{s}/b)}{\max_{1 \leq i \leq m-1} \sum_{j=1}^{m-1} |d_{ij}|} \geq \frac{b + \sqrt{b^2 + 4\|c\|^2}}{4d_{(m-1)(m-1)}} = \frac{1}{4} \left( \text{HG}_1(A) + \sqrt{\text{HG}_1^2(A) + \frac{4\|c\|^2}{d_{(m-1)(m-1)}^2}} \right).$$
■

Because

$$\frac{1}{4} \left( \text{HG}_1(A) + \sqrt{\text{HG}_1^2(A) + \frac{4\|c\|^2}{d_{(m-1)(m-1)}^2}} \right) \geq \frac{1}{2} \text{HG}_1(A),$$

thus, Theorem 3 provides a better lower bound for  $\text{EG}_1(Q)$  than Theorem 1.

Next we turn to give an estimate of the upper bound of  $\text{EG}_1(Q)$ .

**THEOREM 4** *Let  $A \in \mathbb{R}^{n \times m}$  be a hub-dominant matrix having  $a_1, a_2, \dots, a_m$  as its columns with  $\|a_1\| \leq \|a_2\| \leq \dots \leq \|a_m\|$ . Let  $Q = A^t A \in \mathbb{R}^{m \times m}$  in (2) be the corresponding system matrix. Then*

$$\text{EG}_1(Q) \leq \frac{\min\{b + s_1, f_+(s_1/b)\}}{\max\{s_2, f_-(s_3/b)\}}, \quad (15)$$

where

$$\begin{aligned} s_1 &:= \max_{1 \leq i \leq m-1} \sum_{j=1}^{m-1} |d_{ij}|, \\ s_2 &:= \max_{1 \leq i < j \leq m-1} \frac{1}{2} \left( (d_{ii} + d_{jj}) - \sqrt{(d_{ii} - d_{jj})^2 + 4d_{ij}^2} \right), \\ s_3 &:= \max_{1 \leq i < j \leq m-1} \frac{1}{2} \left( (d_{ii} + d_{jj}) + \sqrt{(d_{ii} - d_{jj})^2 + 4d_{ij}^2} \right). \end{aligned}$$

*Proof* Since  $EG_1(Q)$  is the ratio of  $\lambda_m(Q)$  and  $\lambda_{m-1}(Q)$ , we need to give an upper bound for  $\lambda_m(Q)$  and a lower bound for  $\lambda_{m-1}(Q)$ , respectively.

By Lemma 1, we know  $\lambda_m(Q) \leq f_+(\mu_{m-1}/b)$ . By Lemma 6, we get  $\mu_{m-1} \leq s_1$ . The monotonically increasing property of  $f_+$  implies

$$\lambda_m(Q) \leq f_+(s_1/b). \quad (16)$$

On the other hand, by Lemma 7 we obtain another upper bound

$$\lambda_m(Q) \leq b + s_1. \quad (17)$$

Hence, combining (16) and (17) together yields

$$\lambda_m(Q) \leq \min\{b + s_1, f_+(s_1/b)\}. \quad (18)$$

Now, we develop the lower bound for  $\lambda_{m-1}(Q)$ . From Lemmas 3–5, and the comments following Lemma 4, we know that

$$\mu_{m-2} \geq s_2 \quad \text{and} \quad \mu_{m-1} \geq s_3. \quad (19)$$

By Lemma 1 and the monotonically increasing property of  $f_-$ , we have

$$\lambda_{m-1}(Q) \geq \max\{\mu_{m-2}, f_-(\mu_{m-1}/b)\} \geq \max\{s_2, f_-(s_3/b)\}. \quad (20)$$

This completes the proof. ■

*Remark 4* When  $A$  is a hub matrix, Equation (9) implies that  $\min\{b + s_1, f_+(s_1/b)\} = f_+(s_1/b)$ . Therefore, the upper bound for  $EG_1(Q)$  given by Theorem 4 is the same as the one given by Theorem 2.

**COROLLARY 3** *Under the same conditions as in Theorem 4, the upper bound for  $EG_1(Q)$  provided by Theorem 4 is tighter than the one given by Theorem 1.*

*Proof* By Lemma 5, we have

$$d_{(m-2)(m-2)} - \sigma_{m-2} \leq s_2, \quad (21)$$

where  $\sigma_i := \sum_{j=1, j \neq i}^{m-1} |d_{ij}|$  is the sum of the magnitudes of the off-diagonal elements of the  $i$ -th row of  $D$  in (2). The difference  $d_{(m-2)(m-2)} - \sigma_{m-2}$  in above inequality should be understood as the second largest number among  $d_{11} - \sigma_1, d_{22} - \sigma_2, \dots, d_{(m-1)(m-1)} - \sigma_{m-1}$ .

Next, we will show that

$$s_1 \leq d_{(m-1)(m-1)} + \sum_{j=1}^{m-2} \sigma_j. \quad (22)$$

We consider two different cases. In the first case, we assume that  $\sum_{j=1}^{m-1} |d_{(m-1)j}| = s_1$ . Then

$$s_1 = d_{(m-1)(m-1)} + \sum_{j=1}^{m-2} |d_{(m-1)j}| = d_{(m-1)(m-1)} + \sum_{j=1}^{m-2} |d_{j(m-1)}| \leq d_{(m-1)(m-1)} + \sum_{j=1}^{m-2} \sigma_j.$$

In the second case, we assume that there is an integer  $i_0$  between 1 and  $m-2$  such that  $\sum_{j=1}^{m-1} |d_{i_0 j}| = s_1$ . Hence,

$$s_1 = d_{i_0 i_0} + \sum_{\substack{j=1 \\ j \neq i_0}}^{m-2} |d_{i_0 j}| + |d_{i_0(m-1)}| \leq d_{(m-1)(m-1)} + \sum_{\substack{j=1 \\ j \neq i_0}}^{m-2} |d_{j i_0}| + \sigma_{i_0} \leq d_{(m-1)(m-1)} + \sum_{\substack{j=1 \\ j \neq i_0}}^{m-2} \sigma_j + \sigma_{i_0}.$$

Therefore, (22) holds.

By (21) and (22), we have

$$\frac{\min\{b + s_1, f_+(s_1/b)\}}{\max\{s_2, f_-(s_3/b)\}} \leq \frac{b + s_1}{s_2} \leq \frac{b + d_{(m-1)(m-1)} + \sum_{j=1}^{m-2} \sigma_j}{d_{(m-2)(m-2)} - \sigma_{m-2}}.$$

This implies that the upper bound for  $\text{EG}_1(Q)$  given by Theorem 4 is tighter than the one given by Theorem 2.  $\blacksquare$

## 5. Hub-dominant matrices from equiangular tight frames

In this section, we focus on designing a class of hub-dominant matrices. We begin the discussion from hub matrices.

Let  $A = [a_1 \ a_2 \ \cdots \ a_m]$  be a hub matrix with  $a_m$  as its hub column. By the definition of hub matrices,  $\sum_{i=1}^{m-1} |\langle a_j, a_i \rangle| = \|a_j\|^2$  for  $j = 1, \dots, m-1$ , i.e.  $\xi = \eta = 1$  in (1). We ask whether we can construct hub-dominant matrices satisfying the following properties:

$$\sum_{i=1}^{m-1} |\langle a_j, a_i \rangle| = \beta \|a_j\|^2, \quad \text{for } j = 1, \dots, m-1, \quad (23)$$

where  $1 < \beta \leq 2$ . The answer to this question is yes. We can construct a class of such hub-dominant matrices from equiangular tight frames. The definition of equiangular tight frames is given as follows.

*Definition 3* The set of unit vectors  $\{v_1, v_2, \dots, v_N\}$ , where  $v_i \in \mathbb{R}^s$ , is called an *equiangular tight frame* (ETF) if it satisfies two conditions: (i) For some nonnegative  $\alpha$ , we have  $|\langle v_k, v_j \rangle| = \alpha$  when  $1 \leq k < j \leq N$  and (ii)  $\sum_{k=1}^N v_k v_k^t = \frac{N}{s} \text{Id}$ , when  $\text{Id}$  is the identity matrix in  $\mathbb{R}^{s \times s}$ .

We remark that the first condition indicates the vectors in the set being equiangular while the second condition indicates the vectors generating a tight frame, that is (see, e.g. [8]),

$$v = \frac{s}{N} \sum_{k=1}^N \langle v, v_k \rangle v_k \quad \text{for all } v \in \mathbb{R}^s. \quad (24)$$

Equiangular tight frames have been studied recently and have found applications in communications and coding theory (see, e.g. [9–12]).

Immediately, the definitions of hub-dominant matrix and ETF lead to the following results.

PROPOSITION 1 Fix  $m \geq 3$  and let  $A = [a_1 \ a_2 \ \cdots \ a_m]$  be a hub-dominant matrix, where  $a_i \in \mathbb{R}^n$  for  $1 \leq i \leq m$ . If the collection of non-hub columns  $a_1, a_2, \dots, a_{m-1}$  of  $A$  forms an ETF, then (i)  $\|a_m\| \geq 1$ ; (ii)  $\beta = (m-2)\alpha + 1$ , where  $\beta$  is the constant in (23) and  $\alpha = |\langle a_i, a_j \rangle|$  for all  $1 \leq i < j \leq m-1$  and (iii)  $\alpha \leq \frac{1}{m-2}$ .

We are particularly interested in the case where  $\beta = 2$ , i.e.  $\alpha = \frac{1}{m-2}$ , under the conditions of Proposition 1. Mercedes-Benz systems, a special type of ETFs, satisfy this additional requirement.

An ETF  $\{b_1, b_2, \dots, b_{n+1}\}$  in  $\mathbb{R}^n$  is called a *Mercedes-Benz system* if

$$\langle b_k, b_j \rangle = -\frac{1}{n} \quad \text{for } k \neq j.$$

LEMMA 8 Let  $\{b_1, b_2, \dots, b_{n+1}\}$  in  $\mathbb{R}^n$  be a Mercedes-Benz system. Then

$$\sum_{i=1}^{n+1} b_i = 0.$$

*Proof* For every  $1 \leq j \leq n+1$

$$\left\langle \sum_{i=1}^{n+1} b_i, b_j \right\rangle = \langle b_j, b_j \rangle + \left\langle \sum_{\substack{i=1 \\ i \neq j}}^{n+1} b_i, b_j \right\rangle = 1 + \sum_{\substack{i=1 \\ i \neq j}}^{n+1} \langle b_i, b_j \rangle = 1 - \frac{n}{n} = 0.$$

By the second condition in the definition of an ETF, we have

$$\sum_{i=1}^{n+1} b_i = \frac{n}{n+1} \sum_{j=1}^{n+1} \left\langle \sum_{i=1}^{n+1} b_i, b_j \right\rangle b_j = 0.$$

This completes the proof. ■

We remark that the concrete constructions of Mercedes-Benz systems are discussed in [13].

LEMMA 9 Fix  $m \geq 3$  and let  $A = [a_1 \ a_2 \ \cdots \ a_m] \in \mathbb{R}^{(m-2) \times m}$  be a hub-dominant matrix with  $a_m$  as its hub column. Let  $Q$  in (2) be the system matrix associated with  $A$ . If the set of the non-hub columns  $a_1, a_2, \dots, a_{m-1}$  of  $A$  forms a Mercedes-Benz system, then  $D \in \mathbb{R}^{(m-1) \times (m-1)}$ , a submatrix of  $Q$  in (2), has a form of

$$D = \frac{m-1}{m-2} \text{Id} - \frac{1}{m-2} ee^t, \quad (25)$$

where  $e := [1, 1, \dots, 1]^t \in \mathbb{R}^{m-1}$ . Further, the eigenvalues of  $D$  are 0 (simple) and  $\frac{m-1}{m-2}$  with multiplicity  $m-2$ .

*Proof* Since the set of the non-hub columns  $a_1, a_2, \dots, a_{m-1}$  of  $A$  forms a Mercedes-Benz system, then  $d_{ij}$ , the  $(i, j)$ -th entry of  $D$  is

$$d_{ij} = \langle a_i, a_j \rangle = \begin{cases} 1 & \text{if } i = j; \\ -\frac{1}{m-2} & \text{if } i \neq j. \end{cases}$$

It leads to (25).

Since  $\frac{1}{m-2}ee^t$  is a rank-1 matrix and has  $\frac{m-1}{m-2}$  as its only non-zero eigenvalue with  $e$  as the corresponding eigenvector, hence the eigenvalues of  $D$  are 0 (simple) and  $\frac{m-1}{m-2}$  with multiplicity  $m-2$ . ■

**THEOREM 5** Fix  $m \geq 3$  and let  $A = [a_1 \ a_2 \ \cdots \ a_m] \in \mathbb{R}^{(m-2) \times m}$  be a hub-dominant matrix with  $a_m$  as its hub column. Let  $Q$  in (2) be the system matrix associated with  $A$ . If the set of the non-hub columns  $a_1, a_2, \dots, a_{m-1}$  of  $A$  forms a Mercedes-Benz system, then the eigenvalues of  $Q$  are 0 with multiplicity 2,  $\frac{m-1}{m-2}$  with multiplicity  $m-3$ , and  $b + \frac{m-1}{m-2}$  (simple).

*Proof* By Lemma 9, there exists an orthogonal matrix  $U \in \mathbb{R}^{(m-1) \times (m-1)}$  such that

$$D = U^t \Gamma U, \quad \text{where } \Gamma = \text{diag}\left(0, \frac{m-1}{m-2}, \dots, \frac{m-1}{m-2}\right).$$

Many orthogonal matrices  $U$  can serve this purpose. Here we simply choose  $U$  to be the  $(m-1) \times (m-1)$  DCT-II matrix whose first row is  $\frac{1}{\sqrt{m-1}}e$  (see [14]). With the help of  $U$ , we have

$$\begin{bmatrix} \Gamma & z \\ z^t & b \end{bmatrix} = \begin{bmatrix} U & \\ & 1 \end{bmatrix} \begin{bmatrix} D & c \\ c^t & b \end{bmatrix} \begin{bmatrix} U^t & \\ & 1 \end{bmatrix},$$

where  $b = \|a_m\|^2$ ,  $c^t = [\langle a_m, a_1 \rangle \ \langle a_m, a_2 \rangle \ \cdots \ \langle a_m, a_{m-1} \rangle]$  and  $z = Uc$ . Write  $z = [z_1 \ z_2 \ \cdots \ z_{m-1}]$ . Then, by Lemma 8,

$$z_1 = \frac{1}{\sqrt{m-1}}e^t c = \frac{1}{\sqrt{m-1}} \sum_{i=1}^{m-1} \langle a_m, a_i \rangle = \frac{1}{\sqrt{m-1}} \left\langle a_m, \sum_{i=1}^{m-1} a_i \right\rangle = 0.$$

By (24),

$$\|z\|^2 = \|Uc\|^2 = \|c\|^2 = \frac{m-1}{m-2} \|a_m\|^2 = \frac{m-1}{m-2} b. \quad (26)$$

Collecting all these results together, we get

$$\begin{aligned} \det\left(\lambda \text{Id} - \begin{bmatrix} \Gamma & z \\ z^t & b \end{bmatrix}\right) &= \lambda \left(\lambda - \frac{m-1}{m-2}\right)^{m-2} (\lambda - b) - \lambda \left(\lambda - \frac{m-1}{m-2}\right)^{m-3} \|z\|^2 \\ &= \lambda^2 \left(\lambda - \frac{m-1}{m-2}\right)^{m-3} \left(\lambda - b - \frac{m-1}{m-2}\right). \end{aligned}$$

This indicates that 0 with multiplicity 2,  $\frac{m-1}{m-2}$  with multiplicity  $m-3$  and  $b + \frac{m-1}{m-2}$  (simple) are the eigenvalues of  $Q$ . This completes the proof. ■

Under the conditions of Theorem 5, we have

$$\text{EG}_1(Q) = 1 + \frac{m-2}{m-1}b.$$

For a fixed  $b$  (i.e. a fixed hub column),  $\text{EG}_1(Q)$  increases when the number of non-hub columns increase and  $\text{EG}_1(Q)$  is bounded by  $1+b$ . For a fixed number of non-hub columns,  $\text{EG}_1(Q)$  is a linear function of  $b$  with the slope  $\frac{m-2}{m-1}$ .

From Theorem 1, the lower bound of  $\text{EG}_1(Q)$  is  $\frac{1}{2}b$ . However, the upper bound cannot be applied because  $d_{(m-2)(m-2)} - \sigma_{m-2} = 0$ .

By a simple calculation, we know that

$$\underline{s} = 0, \quad s_1 = 2, \quad s_2 = \frac{m-3}{m-2}, \quad s_3 = \frac{m-1}{m-2},$$

where  $\underline{s}$ ,  $s_1$ ,  $s_2$  and  $s_3$ , are defined in Theorems 3 and 4. By using (26) and the definition of  $f_{\pm}$ , we have

$$f_+(\underline{s}) = \frac{1}{2} \left( b + \sqrt{b^2 + \frac{4(m-1)}{m-2}b} \right), \quad f_+(s_1/b) \geq b + s_1, \quad f_-(s_3/b) = 0.$$

Then, the lower and upper bounds for  $\text{EG}_1(Q)$  given by Theorems 3 and 4 are the following:

$$\frac{1}{4} \left( b + \sqrt{b^2 + \frac{4(m-1)}{m-2}b} \right) \leq \text{EG}_1(Q) \leq \left( \frac{m-2}{m-3} \right) (2+b).$$

## 6. Conclusion

We have developed improved bounds for the eigengaps of the system matrices associated with hub-dominant matrices. It would be interested in studying the eigenvectors of the system matrices in the future research. Extension of the current results to multi-hub matrices (i.e. two or more hub columns) will be studied as well.

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